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Uniformity in the Strong Convergence of Self-adjoint Operators

CHRIS RORRES

Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104

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1. Introduction

The purpose of this note is to give a consequence of a theorem of Rellich [3, p. 684]. Rellich showed that if a sequence of self-adjoint operators H_n over a Hilbert space $\mathbf{H}$ converges strongly to the self-adjoint operator H in the generalized sense, then the resolutions of the identity $E_n(\lambda)$ of H_n converge strongly to the resolution of the identity $E(\lambda)$ of H for all λ which are not point eigenvalues of H. We will show that the convergence of $E_n(\lambda)\,x$ to $E(\lambda)\,x$ is in fact uniform in λ for all vectors x belonging to the continuous subspace of the operator H. This result is analogous to a theorem in probability theory, sometimes known as Polya's theorem, which states that if a sequence of distribution functions $F_n(x)$ converges weakly to a continuous distribution function $F(x)$ then the convergence is uniform [1, p. 268].

2. Results

Let H_n be a sequence of self-adjoint operators in a Hilbert space $\mathbf{H}$. The sequence converges strongly to the self-adjoint operator H in the generalized sense if the resolvent operators $R_n(\zeta) = (H_n - \zeta I)^{-1}$ converge strongly in the usual sense as bounded operators to the resolvent operator $R(\zeta) = (H - \zeta I)^{-1}$ of H for some (and hence all) nonreal ζ [2, p. 429]. We denote by $E_n(\lambda)$, $-\infty < \lambda < \infty$, the resolutions of the identity of the operators H_n and by $E(\lambda)$ the resolution of the identity of H. The vector $x \in \mathbf{H}$ is said to belong to the continuous subspace of the operator H if $\| E(\lambda)\, x \|^2$ is a continuous real-valued function of λ (equivalently, if $E(\lambda)\, x$ is a continuous vector-valued function of λ). It is known that the continuous subspace of a self-adjoint operator H is a closed subspace which reduces the operator and is perpendicular to all eigenvectors of H [2, p. 515].

We first prove the following lemma which is a slight generalization of the theorem of Rellich mentioned in the introduction.

LEMMA 1. *If the sequence of self-adjoint operators H_n converges strongly to the self-adjoint operator H in the generalized sense and if for a fixed $x \in \mathbf{H}$, $\| E(\lambda) x \|^2$ is continuous at λ_0, then*

$$E_n(\lambda_0) x \to E(\lambda_0) x \qquad as \qquad n \to \infty.$$

Proof. We set $\lambda_0 = 0$ without loss of generality. Under the hypotheses of the above theorem it is shown in [2, p. 432, Eq. 1.12] that the sequence of bounded operators

$$(E_n(0) - E(0)) H(H^2 + I)^{-1}$$

converges strongly to 0 as $n \to \infty$. Hence $E_n(0) u \to E(0) u$ for all u in the range of $H(H^2 + I)^{-1}$. But this range is dense in the closed subspace S perpendicular to the eigenvectors of H corresponding to the (possible) eigenvalue $\lambda_0 = 0$. Since the sequence $E_n(0)$ is uniformly bounded, it follows that $E_n(0) v \to E(0) v$ for all $v \in S$. And since $E(0) x - E(0 -) x = 0$, x lies in S so that $E_n(0) x \to E(0) x$.

We shall also need the following elementary lemma from Hilbert space theory to facilitate the proof of our theorem.

LEMMA 2. *If $E(\lambda)$ is a resolution of the identity and if $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4$, then for all $x \in \mathbf{H}$*

$$\| E(\lambda_3) x - E(\lambda_2) x \| \leqslant \| E(\lambda_4) x - E(\lambda_1) x \| .$$

Our main result is the following:

THEOREM. *Let the sequence of self-adjoint operators H_n converge strongly to the self-adjoint operator H in the generalized sense. Let x be an element of the continuous subspace of H. Then given any $\epsilon > 0$ there exists an $N = N(\epsilon, x)$ such that*

$$\| E_n(\lambda) x - E(\lambda) x \| < \epsilon$$

for all $n \geqslant N$ and all λ.

Proof. By the properties of resolutions of the identity we know that $E(\lambda) x \to 0$ as $\lambda \to -\infty$. Hence there exists a λ' such that

$$\| E(\lambda') x \| < \epsilon/3.$$

By Lemma 1 for this λ' there exists an N' such that

$$\| E_n(\lambda') x - E(\lambda') x \| < \epsilon/3 \qquad \text{for all } n \geqslant N'.$$

Then for all $\lambda \leqslant \lambda'$ and all $n \geqslant N'$, we have

$$\begin{aligned} \| E_n(\lambda) x - E(\lambda) x \| &\leqslant \| E_n(\lambda) x \| + \| E(\lambda) x \| \\ &\leqslant \| E_n(\lambda') x \| + \| E(\lambda') x \| \\ &\leqslant \| E_n(\lambda') x - E(\lambda') x \| + 2 \| E(\lambda') x \| \\ &< \epsilon. \end{aligned} \tag{1}$$

Next, since $E(\lambda) x \to x$ as $\lambda \to +\infty$, there is a λ'' such that

$$\| x - E(\lambda'') x \| < \epsilon/4.$$

And for this λ'' by Lemma 1 there is an N'' such that

$$\| E_n(\lambda'') x - E(\lambda'') x \| < \epsilon/4 \qquad \text{for all } n \geqslant N''.$$

Then for all $\lambda \geqslant \lambda''$ and all $n \geqslant N''$ we have by Lemma 2 and the triangle inequality

$$\begin{aligned} E_n(\lambda) X - E(\lambda) x \| &\leqslant \| E_n(\lambda) x - E_n(\lambda'') x \| + \| E_n(\lambda'') x - E(\lambda'') x \| \\ &\quad + \| E(\lambda'') x - E(\lambda) x \| \\ &\leqslant \| x - E_n(\lambda'') x \| + \| E_n(\lambda'') x - E(\lambda'') x \| \\ &\quad + \| x - E(\lambda'') x \| \\ &\leqslant \| x - E(\lambda'') x \| + 2 \| E_n(\lambda'') x - E(\lambda'') x \| \\ &\quad + \| x - E(\lambda'') x \| \\ &< \epsilon. \end{aligned} \tag{2}$$

Now consider the closed interval $[\lambda', \lambda'']$. Since the vector-valued function $E(\lambda) x$ is continuous in λ, it is uniformly continuous on that interval. Hence there exists a subdivision of the interval

$$\lambda' = \lambda_1 < \lambda_2 < \cdots < \lambda_m = \lambda''$$

such that

$$\| E(\lambda_k) x - E(\lambda_{k-1}) x \| < \epsilon/5$$

for $k = 2, 3, \ldots, m$. For these m values of λ we may find a value N''' such that

$$\| E_n(\lambda_k) x - E(\lambda_k) x \| < \epsilon/5$$

for all $n \geqslant N'''$ and $k = 1, 2, \ldots, m$. Then consider any λ in the interval $[\lambda', \lambda'']$. There is some k' such that

$$\lambda_{k'-1} \leqslant \lambda \leqslant \lambda_{k'}.$$

Hence for all λ in $[\lambda', \lambda'']$ and all $n \geqslant N'''$ we have, by Lemma 2 and the triangle inequality,

$$\begin{aligned}
\| E_n(\lambda) x - E(\lambda) x \| &\leqslant \| E_n(\lambda) x - E_n(\lambda_{k'-1}) x \| + \| E_n(\lambda_{k'-1}) x - E(\lambda_{k'-1}) x \| \\
&\quad + \| E(\lambda_{k'-1}) x - E(\lambda) x \| \\
&< \| E_n(\lambda_{k'}) x - E_n(\lambda_{k'-1}) x \| + \epsilon/5 \\
&\quad + \| E(\lambda_{k'}) x - E(\lambda_{k'-1}) x \| \qquad (3) \\
&< \| E_n(\lambda_{k'}) x - E(\lambda_{k'}) x \| + \| E(\lambda_{k'}) x - E(\lambda_{k'-1}) x \| \\
&\quad + \| E(\lambda_{k'-1}) x - E_n(\lambda_{k'-1}) x \| + 2\epsilon/5 \\
&< \epsilon.
\end{aligned}$$

Finally, setting $N = \max(N', N'', N''')$, we have from (1), (2), and (3) that

$$\| E_n(\lambda) x - E(\lambda) x \| < \epsilon$$

for all $n \geqslant N$ and all λ.

References

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2. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York, 1966.
3. F. Rellich, Störungstheorie der Spektralzerlegung. II, *Math. Ann.* **113** (1937), 677–685.